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Nonlinear Donati compatibility conditions and the intrinsic approach for nonlinearly elastic plates

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Abstract

Linear Donati compatibility conditions guarantee that the components of symmetric tensor fields are those of linearized change of metric or linearized change of curvature tensor fields associated with the displacement vector field arising in a linearly elastic structure when it is subjected to applied forces. These compatibility conditions take the form of variational equations with divergence-free tensor fields as test-functions, by contrast with Saint-Venant compatibility conditions, which take the form of systems of partial differential equations.

In this paper, we identify and justify nonlinear Donati compatibility conditions that apply to a nonlinearly elastic plate modeled by the Kirchhoff-von Kármán-Love theory. These conditions, which to the authors' best knowledge constitute a first example of nonlinear Donati compatibility conditions, in turn allow to recast the classical approach to this nonlinear plate theory, where the unknown is the position of the deformed middle surface of the plate, into the intrinsic approach, where the change of metric and change of curvature tensor fields of the deformed middle surface of the plate are the only unknowns. The intrinsic approach thus provides a direct way to compute the stress resultants and the stress couples inside the deformed plate, often the unknowns of major interest in computational mechanics.

Résumé

Les conditions de compatibilité de Donati linéaires garantissent que les composantes de champs de tenseurs symétriques sont celles de tenseurs linéarisés de changement de métrique ou de changement de courbure, associés à un champ de déplacements apparaissant dans une structure élastique soumise à des forces appliquées. Ces conditions de compatibilité prennent la forme d'équations variationnelles avec des champs de tenseurs à divergence nulle comme fonctions-tests, par contraste avec les conditions de compatibilité de Saint-Venant, qui prennent la forme de systèmes d'équations aux dérivées partielles.

Dans cet article, nous identifions et justifions des conditions de compatibilité de Donati non linéaires, qui s'appliquent à une plaque non linéairement élastique modélisée selon la théorie de Kirchhoff-von Kármán-Love. Ces conditions, qui à la connaissance des auteurs constituent un premier exemple de conditions de compatibilité de Donati non linéaires, permettent ensuite de reformuler l'approche classique de cette théorie non linéaire de plaques sous la forme de l'approche intrinsèque, où les champs de tenseurs de changement de métrique et de changement de courbure de la surface moyenne déformée de la plaque sont les seules inconnues. L'approche intrinsèque fournit ainsi un moyen direct de calculer les efforts tranchants et les moments fléchissants à l'intérieur de la plaque déformée, souvent les inconnues les plus significatives en calcul des structures.

Keywords: Nonlinear plate theory, Donati compatibility conditions, intrinsic approach

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1. Introduction

Throughout this paper, Latin, resp. Greek, indices and exponents vary in the set $\{1, 2, 3\}$, resp. in the set $\{1, 2\}$, save when they are used for indexing sequences, resp. save ν in the notation ∂_ν. The summation convention with respect to repeated indices and exponents is used in conjunction with these rules.

A *domain* ω in $\mathbb{R}^2$ is a bounded and connected open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ, the set ω being locally on the same side of γ. Given a domain $\omega \subset \mathbb{R}^2$, the notations $\partial_\alpha := \partial/\partial y_\alpha$, $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$, etc., designate partial derivatives, possibly in the sense of distributions, of functions of $(y_\alpha) \in \omega$.

We now briefly describe the well-known, and often used, *Kirchhoff-von Kármán-Love theory of a nonlinearly elastic plate*, so named after Kirchhoff [17], von Kármán [22], and Love [18].

Let ω be a domain in $\mathbb{R}^2$. Consider an *elastic plate* of *thickness* $2\varepsilon > 0$ with $\overline{\omega}$ as its middle surface, made up with a homogeneous and isotropic elastic material, and whose reference configuration $\overline{\omega} \times [-\varepsilon, \varepsilon]$ is a natural state. Let

$$a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu}\delta_{\alpha\beta}\delta_{\sigma\tau} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma})$$

denote the components of the *two-dimensional elasticity tensor of the plate*, where $\lambda \geq 0$ and $\mu > 0$ denote the *Lamé constants* of the constituting material of the plate; let $(p_i) \in L^2(\omega; \mathbb{R}^3)$ and $(q_\alpha) \in L^2(\omega; \mathbb{R}^2)$ respectively denote the resultants and couples of the given applied forces. Finally, assume that the plate is *clamped* on a $\mathrm{d}\gamma$-measurable subset γ_0 of $\gamma := \partial\omega$ (note that γ_0 may be empty).

Let the functional J be defined for each vector field $\boldsymbol{\eta} := (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ by

$$J(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega \left\{ \frac{\varepsilon}{4} a_{\alpha\beta\sigma\tau}(\partial_\sigma\eta_\tau + \partial_\tau\eta_\sigma + \partial_\sigma\eta_3\partial_\tau\eta_3)(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) + \frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\eta_3\partial_{\alpha\beta}\eta_3 \right\} \mathrm{d}y - L(\boldsymbol{\eta}),$$

where

$$L(\boldsymbol{\eta}) := \int_\omega p_i\eta_i \,\mathrm{d}y - \int_\omega q_\alpha\partial_\alpha\eta_3 \,\mathrm{d}y,$$

and let the space $\boldsymbol{V}(\omega)$ be defined by

$$\boldsymbol{V}(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\},$$

where ∂_ν designates the outer normal derivative operator along γ (the operator ∂_ν is well-defined $\mathrm{d}\gamma$-almost everywhere along γ, since the unit outer normal vector is itself well-defined $\mathrm{d}\gamma$-almost everywhere along the boundary of a domain).

Then, according to the *Kirchhoff-von Kármán-Love theory of a nonlinearly elastic plate* (see, e.g., Ciarlet [3]), the vector field $\boldsymbol{\eta}^* = (\eta_i^*)$, where η_i^* are the Cartesian components of the displacement vector field of the middle surface $\overline{\omega}$ of the plate, should be the solution of the following minimization problem:

$$\boldsymbol{\eta}^* \in \boldsymbol{V}(\omega) \text{ and } J(\boldsymbol{\eta}^*) = \inf_{\boldsymbol{\eta}\in\boldsymbol{V}(\omega)} J(\boldsymbol{\eta}).$$

If $0 < \mathrm{d}\gamma\text{-meas}\,\gamma_0 \leq \mathrm{d}\gamma\text{-meas}\,\gamma$, this minimization problem has at least one solution if the norms $\|p_\alpha\|_{L^2(\omega)}$ are small enough (see Ciarlet & Destuynder [6]; see also Nečas & Naumann [20] in the special case where $p_\alpha = 0$). If $\gamma_0 = \emptyset$, in which case $\boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, this minimization problem has a solution if (and only if) the components p_i and q_α of the resultants and couples of the applied forces satisfy ad hoc *compatibility conditions*, and if the norms $\|p_\alpha\|_{L^2(\omega)}$ are again small enough (see Ciarlet & S. Mardare [8]); note that the solution is never unique in this case, however (see the discussion given in *ibid*.).

Let $\mathbb{L}^2(\omega)$ denote the space of all 2×2 symmetric tensor fields with components in $L^2(\omega)$. The nonlinear part of the integrand appearing in the functional J is a function of the *change of metric tensor field* $(E_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and of the *change of curvature tensor field* $(F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$, the components of which are respectively defined for any vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$ by

$$E_{\alpha\beta} := \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) \text{ and } F_{\alpha\beta} := \partial_{\alpha\beta}\eta_3$$

(despite their names, which we adopt here because they are commonly used in nonlinear plate theory, these tensors are in effect only *ad hoc* approximations of the "true" change of metric and change of curvature tensors of the middle surface).

By contrast with the *classical approach* described above, where the Cartesian components η_i^* of the displacement field are the unknowns, an *intrinsic approach* to the same problem consists instead in *considering the components*

$$E_{\alpha\beta}^* := \frac{1}{2}(\partial_\alpha\eta_\beta^* + \partial_\beta\eta_\alpha^* + \partial_\alpha\eta_3^*\partial_\beta\eta_3^*) \text{ and } F_{\alpha\beta}^* := \partial_{\alpha\beta}\eta_3^*$$

of the corresponding change of metric and change of curvature tensor fields as the unknowns. The intrinsic approach thus provides a direct way to compute the *stress resultants* $n_{\alpha\beta}$ and the *stress couples* $m_{\alpha\beta}$ inside the plate since these are respectively defined as

$$n_{\alpha\beta} := a_{\alpha\beta\sigma\tau}E_{\sigma\tau}^* \text{ and } m_{\alpha\beta} := a_{\alpha\beta\sigma\tau}F_{\sigma\tau}^*.$$

This feature constitutes an advantage of the intrinsic approach over the classical approach, inasmuch as the stress resultants $n_{\alpha\beta}$ and the stress couples $m_{\alpha\beta}$ are often considered to be the unknowns of interest (rather than the components η_i^* of the displacement vector field) in the computation of elastic structures; in this direction, see notably the pioneering contributions of W. Pietraszkiewicz and his school [21].

In order to recast the minimization problem of the classical approach into one of the intrinsic approach, the first objective thus consists in finding necessary and sufficient conditions guaranteeing that, given two tensor fields $(E_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and $(F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$, there exists a vector field $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$ such that

$$\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) = E_{\alpha\beta} \text{ and } \partial_{\alpha\beta}\eta_3 = F_{\alpha\beta}.$$

A first answer to this question when $\gamma_0 = \emptyset$, in which case $\boldsymbol{V}(\omega) = H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, was recently given in Ciarlet & S. Mardare [8]. There, it was shown that, if the open set ω is *simply-connected*, a first class of such necessary and sufficient conditions can be expressed as a *system of nonlinear partial differential equations* in the sense of distributions, which take the form of the following *nonlinear Saint-Venant compatibility conditions*:

$$\partial_{\sigma\tau}E_{\alpha\beta} + \partial_{\alpha\beta}E_{\sigma\tau} - \partial_{\alpha\sigma}E_{\beta\tau} - \partial_{\beta\tau}E_{\alpha\sigma} = F_{\alpha\sigma}F_{\beta\tau} - F_{\alpha\beta}F_{\sigma\tau} \text{ in } H^{-2}(\omega),$$
$$\partial_\sigma F_{\alpha\beta} = \partial_\beta F_{\alpha\sigma} \text{ in } H^{-1}(\omega).$$

In this paper, we show that, *when* $\gamma_0 = \gamma$, in which case $\boldsymbol{V}(\omega) = H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$, *necessary and sufficient nonlinear compatibility conditions "of Donati type"* can be found (Theorem 3.1), which this time take the form of *nonlinear variational equations, with divergence-free tensor fields as test-functions*; note that, at least to the authors' best knowledge, these constitute a first example of nonlinear compatibility conditions of Donati type.

Once such compatibility conditions are identified and justified, they in turn allow to recast the corresponding minimization problem into one of the *intrinsic approach* (Theorem 4.2)

Some of the results of this paper were announced (mostly without proofs) in Ciarlet, Geymonat & Krasucki [7].

2. Technical preliminaries

This section gathers various preliminary results that will all be used in Section 3 for identifying the nonlinear Donati compatibility conditions just alluded to.

Given a domain $\omega \subset \mathbb{R}^2$, the notations $H^m(\omega), H^{-m}(\omega)$, and $H_0^m(\omega)$, $m \geq 1$, designate the usual Sobolev spaces and their dual spaces, the notations $H^{1/2}(\tilde{\gamma})$ and $H^{-1/2}(\tilde{\gamma})$ designate the usual trace space and its dual space over any connected component $\tilde{\gamma}$ of the boundary of ω, the notation $\mathcal{D}(\omega)$ designates the space of infinitely differentiable functions with compact support in ω, and $\mathcal{D}'(\omega)$ designates the space of distributions in ω.

Vector and matrix fields are designated by boldface letters. Spaces of vector, resp. symmetric matrix, fields are designated by boldface, resp. special Roman, letters.

Given a normed vector space V, the notation V' designates its dual space, and ${}_{V'}\langle\cdot,\cdot\rangle_V$ designates the duality between V and V'. If W is a subspace of V, the notation $W \hookrightarrow V$, resp. $W \Subset V$, means that the canonical injection from W into V is continuous, resp. compact.

The following *linear Saint-Venant compatibility conditions* constitute an extension of a classical result for smooth functions to distributions.

Theorem 2.1. *Let there be given a simply-connected domain $\omega \subset \mathbb{R}^2$ and a symmetric matrix field $(F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$. Then there exists a function $w \in H^2(\omega)$ such that*

$$\partial_{\alpha\beta} w = F_{\alpha\beta} \text{ in } L^2(\omega),$$

if and only if

$$\partial_\sigma F_{\alpha\beta} - \partial_\beta F_{\alpha\sigma} = 0 \text{ in } H^{-1}(\omega).$$

If this is the case, any other function $\tilde{w} \in H^2(\omega)$ such that $\partial_{\alpha\beta}\tilde{w} = F_{\alpha\beta}$ in $L^2(\omega)$ is of the form

$$\tilde{w}(y) = w(y) + \alpha_0 + \alpha_1 y_1 + \alpha_2 y_2 \text{ for almost all } y = (y_\alpha) \in \omega,$$

for some constants a_0, α_1, and α_2.

Proof. It is clear that, given any domain $\omega \subset \mathbb{R}^2$ (i.e., simply-connected or not) and given any function $w \in H^2(\omega)$, the matrix field $(F_{\alpha\beta}) := (\partial_{\alpha\beta} w) \in \mathbb{L}^2(\omega)$ satisfies $\partial_\sigma F_{\alpha\beta} = \partial_\beta F_{\alpha\sigma}$ in $H^{-1}(\omega)$.

To establish the converse property, we first observe that all the relations $\partial_\sigma F_{\alpha\beta} - \partial_\beta F_{\alpha\sigma} = 0$ in $H^{-1}(\omega)$ (i.e., for all $\alpha, \beta, \sigma \in \{1, 2\}$) are satisfied if (and only if)

$$\partial_1 F_{12} - \partial_2 F_{11} = 0 \text{ and } \partial_1 F_{22} - \partial_2 F_{21} = 0.$$

The *weak Poincaré lemma* of the form established in Ciarlet & Ciarlet, Jr. [5] (a simpler proof was subsequently given by Kesavan [16]; see also Theorem 6.17-4 in Ciarlet [4]) asserts that, given any vector field $(h_\alpha) \in H^{-1}(\omega)$ that satisfies

$$\partial_\alpha h_\beta - \partial_\beta h_\alpha = 0 \text{ in } H^{-2}(\omega),$$

there exists a function $\theta \in L^2(\omega)$, unique up to the addition of a constant function, such that

$$\partial_\alpha \theta = h_\alpha \text{ in } H^{-1}(\omega).$$

Note that the assumption of *simple-connectedness* of the domain ω is an essential assumption in this lemma, as its proof relies on the "classical" Poincaré lemma (i.e., for smooth functions).

Since

$$\partial_1 F_{12} - \partial_2 F_{11} = 0, \text{ resp. } \partial_1 F_{22} - \partial_2 F_{21} = 0, \text{ in } H^{-1}(\omega),$$

there thus exists a function $\theta \in H^1(\omega)$, resp. $\chi \in H^1(\omega)$, such that

$$\partial_1\theta = F_{11} \text{ and } \partial_2\theta = F_{12}, \text{ resp. } \partial_1\chi = F_{21} \text{ and } \partial_2\chi = F_{22}, \text{ in } L^2(\omega).$$

Since then

$$\partial_1\chi - \partial_2\theta = F_{21} - F_{12} = 0 \text{ in } L^2(\omega),$$

another application of the same lemma shows that there exists a function $w \in H^2(\omega)$ such that

$$\partial_1 w = \theta \text{ and } \partial_2 w = \chi \text{ in } H^1(\omega),$$

hence such that

$$\partial_{11} w = \partial_1\theta = F_{11},\ \partial_{12} w = \partial_1\chi = F_{21},\ \partial_{21} w = \partial_2\theta = F_{12},\ \partial_{22} w = \partial_2\chi = F_{22} \text{ in } L^2(\omega).$$

Such a function w is unique up to the addition of a polynomial of degree ≤ 1 in the variable $y = (y_\alpha)$ since any distribution $T \in \mathcal{D}'(\omega)$ satisfying $\partial_{\alpha\beta} T = 0$ in ω is a polynomial of degree ≤ 1 in y (recall that a domain is connected by assumption). □

A simple re-writing of the compatibility conditions of Theorem 2.1 shows that the function w found in this theorem can be also viewed as an *Airy function*. Recall that the divergence of a 2×2 tensor field $\boldsymbol{S} = (S_{\alpha\beta})$ defined over a two-dimensional open set is the vector field $\mathbf{div}\,\boldsymbol{S} := (\partial_\beta S_{\alpha\beta})$.

Theorem 2.2. *Let there be given a simply-connected domain $\omega \subset \mathbb{R}^2$ and a symmetric tensor field $\boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega)$.*
Then there exists a function $w \in H^2(\omega)$ such that

$$\partial_{11} w = S_{22}, \quad \partial_{12} w = -S_{12}, \quad \partial_{22} w = S_{11} \text{ in } L^2(\omega),$$

if and only if

$$\mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega).$$

If this is the case, the function w is uniquely determined up to the addition of a polynomial of degree ≤ 1 in the variable $y = (y_\alpha)$.

Proof. The relations $\partial_1 F_{12} - \partial_2 F_{11} = 0$ and $\partial_1 F_{22} - \partial_2 F_{21} = 0$ in $H^{-1}(\omega)$ used in the proof of Theorem 2.1 can be equivalently re-written as

$$\mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega), \text{ where } \boldsymbol{S} = (S_{\alpha\beta}) := \begin{pmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{pmatrix} \in \mathbb{L}^2(\omega).$$

Hence the conclusions immediately follow from Theorem 2.1. ☐

Given a symmetric matrix field $(S_{\alpha\beta})$ defined over ω, any function w such that $\partial_{11} w = S_{22}$, $\partial_{12} w = -S_{12}$, and $\partial_{22} w = S_{11}$ is called an **Airy function for the field** $(S_{\alpha\beta})$.

When the domain ω is not simply-connected, an Airy function still exists, provided the compatibility condition $\mathbf{div}\, \boldsymbol{S} = \mathbf{0}$ in $\boldsymbol{H}^{-1}(\omega)$ found in Theorem 2.2 is complemented by other conditions, according to the following result due to Geymonat & Krasucki [11], which constitutes a weak version of a classical result for smooth functions (see, e.g., Ciarlet & Rabier [9]).

Theorem 2.3. *Let $\omega \subset \mathbb{R}^2$ be a non-simply connected domain whose boundary γ consists of $q \geq 2$ connected components γ_i, $1 \leq i \leq q$, let (ν_α) denote the unit outer normal vector along $\gamma = \bigcup_{i=1}^q \gamma_i$, and let the functions $p_0, p_1, p_2 : \overline{\omega} \to \mathbb{R}$ be respectively defined by $p_0(y) = 1$, $p_1(y) = y_1$, $p_2(y) = y_2$ for each $y = (y_\alpha) \in \overline{\omega}$.*
Let $\boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ be a symmetric tensor field. Then there exists a function $w \in H^2(\omega)$ such that

$$\partial_{11} w = S_{22}, \quad \partial_{12} w = -S_{12}, \quad \partial_{22} w = S_{11} \text{ in } L^2(\omega),$$

if and only if

$$\begin{aligned} &\mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega), \\ &{}_{H^{-1/2}(\gamma_i)}\langle S_{\alpha\beta}\nu_\beta, p_0 \rangle_{H^{1/2}(\gamma_i)} = 0, \ 1 \leq i \leq q, \\ &{}_{H^{-1/2}(\gamma_i)}\langle S_{1\beta}\nu_\beta, p_2 \rangle_{H^{1/2}(\gamma_i)} = {}_{H^{-1/2}(\gamma_i)}\langle S_{2\beta}\nu_\beta, p_1 \rangle_{H^{1/2}(\gamma_i)}, \ 1 \leq i \leq q. \end{aligned}$$

If this is the case, the function w is uniquely determined up to the addition of a polynomial of degree ≤ 1 in the variable y.

Note that duality brackets such as ${}_{H^{-1/2}(\gamma_i)}\langle S_{\alpha\beta}\nu_\beta, p_0 \rangle_{H^{1/2}(\gamma_i)}$ are indeed well-defined, since any tensor field $\boldsymbol{S} \in \mathbb{L}^2(\omega)$ satisfying $\mathbf{div}\, \boldsymbol{S} = \mathbf{0}$ in $\boldsymbol{H}^{-1}(\omega)$ belongs to the space $\mathbb{H}(\mathbf{div}; \omega)$; consequently, each "restriction of $S_{\alpha\beta}\nu_\beta$ to γ_i" is well defined as a distribution in the space $H^{-1/2}(\gamma_i)$ (see, e.g., Girault & Raviart [15] or Brezzi & Fortin [2]).

The compatibility conditions guaranteeing that the components $F_{\alpha\beta}$ of a symmetric 2×2 matrix field in $\mathbb{L}^2(\omega)$ can be written as $F_{\alpha\beta} = \partial_{\alpha\beta} w$ for some function $w \in H^2(\omega)$ were identified in Theorem 2.1 as a system of *partial differential equations*. We now show (Theorem 2.4(a)) that it is possible to find linear compatibility conditions, again bearing on the functions $F_{\alpha\beta}$, of a *completely different nature* that achieve the same purpose. These different compatibility conditions are of *Donati type*, in the sense that they take the form of *variational equations* to be satisfied by *divergence-free tensor fields*. We also identify (Theorem 2.4(b)) another class of linear compatibility conditions of Donati type that will be used in the sequel.

Note that, by contrast with the compatibility conditions of Theorem 2.1, those of Theorem 2.4 below no longer require that the domain ω be simply-connected.

Theorem 2.4. (a) *Let there be given a domain $\omega \subset \mathbb{R}^2$ and a tensor field $(F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$.*
Then there exists a function $\eta \in H_0^2(\omega)$ such that

$$\partial_{\alpha\beta}\eta = F_{\alpha\beta} \text{ in } L^2(\omega),$$

if and only if

$$\int_\omega F_{\alpha\beta}T_{\alpha\beta}\,\mathrm{d}y = 0 \text{ for all } (T_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ such that } \operatorname{div}\mathbf{div}\,\boldsymbol{T} := \partial_{\alpha\beta}T_{\alpha\beta} = 0 \text{ in } H^{-2}(\omega).$$

If this is the case, the function η is uniquely determined.
(b) *Let there be given a domain $\dot{\omega} \subset \mathbb{R}^2$ and a tensor field $(e_{\alpha\beta}) \in \mathbb{L}^2(\omega)$.*
Then there exists a vector field $(\eta_\alpha) \in H_0^1(\omega) \times H_0^1(\omega)$ such that

$$\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) = e_{\alpha\beta} \text{ in } L^2(\omega),$$

if and only if

$$\int_\omega e_{\alpha\beta}s_{\alpha\beta}\,\mathrm{d}y = 0 \text{ for all } \boldsymbol{S} = (s_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ such that } \mathbf{div}\,\boldsymbol{S} = (\partial_\beta s_{\alpha\beta}) = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega).$$

If this is the case, the vector field (η_α) is uniquely determined.

Proof. Assume first that $F_{\alpha\beta} = \partial_{\alpha\beta}\eta$ for some $\eta \in H_0^2(\omega)$, so that

$$\int_\omega F_{\alpha\beta}T_{\alpha\beta}\,\mathrm{d}y = {}_{L^2(\omega)}\langle T_{\alpha\beta}, \partial_{\alpha\beta}\eta\rangle_{L^2(\omega)} = {}_{H^{-2}(\omega)}\langle \partial_{\alpha\beta}T_{\alpha\beta}, \eta\rangle_{H_0^2(\omega)}$$

for each tensor field $(T_{\alpha\beta}) \in \mathbb{L}^2(\omega)$. Hence $\int_\omega F_{\alpha\beta}T_{\alpha\beta}\,\mathrm{d}y = 0$ if $\partial_{\alpha\beta}T_{\alpha\beta} = 0$ in $H^{-2}(\omega)$. This proves the “only if” part of Theorem 2.4(a).

Next, let

$$\boldsymbol{A}\eta := \begin{pmatrix} \partial_{11}\eta & \partial_{12}\eta \\ \partial_{21}\eta & \partial_{22}\eta \end{pmatrix} \in \mathbb{L}^2(\omega) \text{ for each } \eta \in H_0^2(\omega).$$

Then the linear operator $\boldsymbol{A} : H_0^2(\omega) \to \mathbb{L}^2(\omega)$ defined in this fashion is clearly continuous. Besides, since the semi-norm $\eta \in\!\to \left(\sum_{\alpha,\beta} \left\|\partial_{\alpha\beta}\eta\right\|^2_{L^2(\omega)}\right)^{1/2}$ is a norm over the space $H_0^2(\omega)$ which is equivalent to the norm $\|\cdot\|_{H^2(\omega)}$ over this space, there exists a constant C such that

$$\|\eta\|_{H^2(\omega)} \le C\,\|\boldsymbol{A}\eta\|_{\mathbb{L}^2(\omega)} \text{ for all } \eta \in H_0^2(\omega).$$

Consequently, the image $\mathbf{Im}\,\boldsymbol{A}$ of $H_0^2(\omega)$ under $\boldsymbol{A}$ is *closed* in $\mathbb{L}^2(\omega)$.

For any matrix field $\boldsymbol{T} = (T_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and any function $\eta \in H_0^2(\omega)$, we have

$$\begin{aligned} {}_{\mathbb{L}^2(\omega)}\langle \boldsymbol{T}, \boldsymbol{A}\eta\rangle_{\mathbb{L}^2(\omega)} &:= {}_{L^2(\omega)}\langle T_{\alpha\beta}, \partial_{\alpha\beta}\eta\rangle_{L^2(\omega)} = {}_{H^{-2}(\omega)}\langle \partial_{\alpha\beta}T_{\alpha\beta}, \eta\rangle_{H_0^2(\omega)} \\ &= {}_{H^{-2}(\omega)}\langle \operatorname{div}\mathbf{div}\,\boldsymbol{T}, \eta\rangle_{H_0^2(\omega)}, \end{aligned}$$

which shows that the *dual operator* of $\boldsymbol{A} : H_0^2(\omega) \to \mathbb{L}^2(\omega)$ is $\operatorname{div}\mathbf{div} : \mathbb{L}^2(\omega) \to H^{-2}(\omega)$.

Banach closed range theorem therefore implies that

$$\operatorname{Im}\boldsymbol{A} = \{\boldsymbol{F} \in \mathbb{L}^2(\omega);\ {}_{\mathbb{L}^2(\omega)}\langle \boldsymbol{F}, \boldsymbol{T}\rangle_{\mathbb{L}^2(\omega)} = 0 \text{ for all } \boldsymbol{T} \in \operatorname{Ker}(\operatorname{div}\mathbf{div})\},$$

which is exactly what the “if part” of Theorem 2.4(a) asserts.

That Ker $\boldsymbol{A} = \{0\}$ implies that the function denoted η is uniquely determined. This proves (a).

The proof of (b) is well-known; see, e.g. Geymonat & Suquet [14], Geymonat & Krasucki [12, 13], or Amrouche, Ciarlet, Gratie & Kesavan [1]. ☐

Finally, we establish a specific *Green's formula*.

Theorem 2.5. *Let ω be a domain in $\mathbb{R}^2$. Then, for any functions $\eta \in H_0^2(\omega)$ and $w \in H^2(\omega)$*

$$\int_\omega (\partial_{11}\eta\partial_{22}\eta - \partial_{12}\eta\partial_{12}\eta) w \, dy = \int_\omega \left\{ -\frac{1}{2}(\partial_1\eta)^2\partial_{22}w - \frac{1}{2}(\partial_2\eta)^2\partial_{11}w + \partial_1\eta\partial_2\eta\partial_{12}w \right\} dy.$$

Proof. Considered as functions of $(\eta, w) \in H_0^2(\omega) \times H^2(\omega)$, both sides of the above relation are continuous (the space $H^2(\omega)$ is continuously imbedded in the space $W^{1,4}(\omega)$ since ω is a two-dimensional domain; this shows that the trilinear form in the right-hand side of this relation is continuous). Since $\mathcal{D}(\omega)$ is dense in $H_0^2(\omega)$ it thus suffices to establish the Green's formula for functions $\eta \in \mathcal{D}(\omega)$ and $w \in H^2(\omega)$. Using the usual formulas of Sobolev spaces (which is licit since ω is assumed to be a domain; cf., e.g., Nečas [19]) and noting that all the boundary integrals vanish in these formulas if $\eta \in \mathcal{D}(\omega)$, we obtain, for any $\eta \in \mathcal{D}(\omega)$ and $w \in H^2(\omega)$,

$$\begin{aligned}
\frac{1}{2}\int_\omega (\partial_1\eta)^2\partial_{22}w \, dy &= -\frac{1}{2}\int_\omega \left[\partial_2(\partial_1\eta)^2\right]\partial_2 w \, dy = -\int_\omega \partial_1\eta\partial_{12}\eta\partial_2 w \, dy \\
&= \int_\omega [\partial_2(\partial_1\eta\partial_{12}\eta)] \, w \, dy \\
&= \int_\omega (\partial_{12}\eta\partial_{12}\eta) w \, dy + \int_\omega (\partial_1\eta\partial_{122}\eta) w \, dy, \\
\frac{1}{2}\int_\omega (\partial_2\eta)^2\partial_{11}w \, dy &= \int_\omega (\partial_{12}\eta\partial_{12}\eta) w \, dy + \int_\omega (\partial_2\eta\partial_{112}\eta) w \, dy, \\
\int_\omega \partial_1\eta\partial_2\eta\partial_{12}w \, dy &= -\int_\omega \partial_1(\partial_1\eta\partial_2\eta)\partial_2 w \, dy = -\int_\omega (\partial_{11}\eta\partial_2\eta + \partial_1\eta\partial_{12}\eta)\partial_2 w \, dy \\
&= \int_\omega [\partial_2(\partial_{11}\eta\partial_2\eta) + \partial_2(\partial_1\eta\partial_{12}\eta)] \, w \, dy \\
&= \int_\omega [\partial_{112}\eta\partial_2\eta + \partial_{11}\eta\partial_{22}\eta + \partial_{12}\eta\partial_{12}\eta + \partial_1\eta\partial_{122}\eta] \, w \, dy,
\end{aligned}$$

from which the announced Green's formula follows. □

3. Nonlinear Donati compatibility conditions

The following theorem constitutes the *first main result* of this paper. For brevity, it is stated and established for a simply-connected domain ω, but it should be clear that a similar result (based on Theorem 2.3 instead of on Theorem 2.2) holds if ω is not simply-connected.

Theorem 3.1. *Let ω be a simply-connected domain in $\mathbb{R}^2$. Given a matrix field $\boldsymbol{S} \in \mathbb{L}^2(\omega)$ that satisfies*

$$\mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega),$$

there exists a unique function $w \in H^2(\omega)$ such that

$$\partial_{11}w = S_{22}, \quad \partial_{12}w = -S_{12}, \quad \partial_{22}w = S_{11} \text{ in } L^2(\omega), \text{ and } \int_\omega w \, dy = \int_\omega \partial_\alpha w \, dy = 0.$$

Let

$$\Phi := \left\{ \boldsymbol{S} \in \mathbb{L}^2(\omega);\ \mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega) \right\} \to \left\{ \psi \in H^2(\omega);\ \int_\omega \psi \, dy = \int_\omega \partial_\alpha\psi \, dy = 0 \right\}$$

denote the mapping defined in this fashion, i.e., by $\Phi(\boldsymbol{S}) := w$.

Let there be given two matrix fields $(E_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and $(F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$. Then there exists a vector field $(\eta_1, \eta_2, \eta_3) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ such that

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) = E_{\alpha\beta} \text{ in } L^2(\omega),$$
$$\partial_{\alpha\beta}\eta_3 = F_{\alpha\beta} \text{ in } L^2(\omega),$$

if and only if the following **nonlinear Donati compatibility conditions** *are satisfied:*

$$\int_\omega F_{\alpha\beta} T_{\alpha\beta}\, dy = 0 \text{ for all } \boldsymbol{T} = (T_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ such that } \operatorname{div} \mathbf{div}\, \boldsymbol{T} = 0 \text{ in } H^{-2}(\omega),$$
$$\int_\omega \{E_{\alpha\beta} S_{\alpha\beta} + (\det \boldsymbol{F})\Phi(\boldsymbol{S})\}\, dy = 0 \text{ for all } \boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ such that } \mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega).$$

If this is the case, such a vector field (η_1, η_2, η_3) is uniquely determined.

Proof. The function w found in Theorem 2.2 is unique up to the addition of a polynomial of degree ≤ 1 in y; hence w becomes uniquely determined if it is subjected to satisfy in addition the relations $\int_\omega w\, dy = \int_\omega \partial_\alpha w\, dy = 0$. This shows that the mapping Φ is well-defined.

So, let two tensor fields $(E_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and $(F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ be given that satisfy the above nonlinear Donati compatibility conditions. We then first infer from Theorem 2.4(a) that there exists a uniquely determined function $\eta_3 \in H_0^2(\omega)$ such that

$$F_{\alpha\beta} = \partial_{\alpha\beta}\eta_3 \text{ in } L^2(\omega).$$

Second, let $\boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ be a tensor field that satisfies $\mathbf{div}\, \boldsymbol{S} = 0$ in $\boldsymbol{H}^{-1}(\omega)$. Hence there exists by Theorem 2.2 one and only one function $w \in H^2(\omega)$ such that

$$S_{11} = \partial_{22} w, \quad S_{12} = -\partial_{12} w, \quad S_{22} = \partial_{11} w \text{ in } L^2(\omega),$$
$$\int_\omega w\, dy = \int_\omega \partial_\alpha w\, dy = 0.$$

Consequently, for any such tensor field $\boldsymbol{S}$,

$$\int_\omega E_{\alpha\beta} S_{\alpha\beta}\, dy = \int_\omega \{E_{11}\partial_{22} w + E_{22}\partial_{11} w - 2E_{12}\partial_{12} w\}\, dy,$$

on the one hand, and

$$\int_\omega (\det \boldsymbol{F})\Phi(\boldsymbol{S})\, dy = \int_\omega \left(F_{11}F_{22} - (F_{12})^2\right) w\, dy$$
$$= \int_\omega (\partial_{11}\eta_3 \partial_{22}\eta_3 - \partial_{12}\eta_3 \partial_{22}\eta_3) w\, dy,$$

on the other hand. Using the Green's formula of Theorem 2.5, we can therefore re-write the left-hand side of the second Donati compatibility condition as

$$\begin{aligned}
&\int_\omega \left\{E_{\alpha\beta} S_{\alpha\beta} + \det(\boldsymbol{F})\boldsymbol{\Phi}(\boldsymbol{S})\right\} dy \\
&= \int_\omega \left\{ \left(E_{11} - \frac{1}{2}(\partial_1\eta_3)^2\right)\partial_{22} w - 2\left(E_{12} - \frac{1}{2}\partial_1\eta_3\partial_2\eta_3\right)\partial_{12} w \right. \\
&\qquad \left. + \left(E_{22} - \frac{1}{2}(\partial_2\eta_3)^2\right)\partial_{11} w \right\} dy \\
&= \int_\omega \left\{ \left(E_{11} - \frac{1}{2}(\partial_1\eta_3)^2\right) S_{11} + 2\left(E_{12} - \frac{1}{2}\partial_1\eta_3\partial_2\eta_3\right) S_{12} \right. \\
&\qquad \left. + \left(E_{22} - \frac{1}{2}(\partial_2\eta_3)^2\right) S_{22} \right\} dy.
\end{aligned}$$

Since this last relation holds for all $\boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ such that $\partial_\beta S_{\alpha\beta} = 0$ in $H^{-1}(\omega)$, there exists a uniquely determined vector field $(\eta_\alpha) \in H_0^1(\omega) \times H_0^1(\omega)$ such that (Theorem 2.4(b))

$$E_{\alpha\beta} - \frac{1}{2}\partial_\alpha\eta_3\partial_\beta\eta_3 = \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) \text{ in } L^2(\omega).$$

This completes the proof of the "if part".

The "only if part" follows from Theorem 2.4(a) for the first Donati compatibility conditions and by reversing the above computations for the second one. □

Remark As expected, the "linearization" of the nonlinear Donati compatibility conditions found in Theorem 3.1, which simply consists in deleting the nonlinear term $\int_\omega (\det \boldsymbol{F})\Phi(\boldsymbol{S})\, dy$, reduces to the linear Donati compatibility conditions of Theorem 2.4. □

4. The intrinsic approach to nonlinear plate theory by means of Donati compatibility conditions

Our analysis of the intrinsic approach applied to the minimization problem of Section 1 in the special case where $\gamma_0 = \gamma$ is essentially based on the properties of a specific *set* $\mathbb{T}(\omega)$ *of admissible tensor fields* and of a specific *nonlinear mapping acting from* $\mathbb{T}(\omega)$ *onto the space* $H_0^1(\omega) \times H_0^1(\omega) \times H_0^1(\omega)$, the definitions and properties of which are the object of the next theorem. For simplicity, we again consider the case where the domain ω is simply-connected, but the extension to general domains (based on Theorem 2.3 instead of Theorem 2.2) is clearly possible.

Since the proof of the next theorem is similar to that of Theorem 5.1 in Ciarlet & S. Mardare [8], it is only sketched.

Theorem 4.1. *Let ω be a simply-connected domain in $\mathbb{R}^2$.*

(a) *Define the set*

$$\begin{aligned}\mathbb{T}(\omega) := \{&((E_{\alpha\beta}, (F_{\alpha\beta})) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega);\\ &\int_\omega F_{\alpha\beta}T_{\alpha\beta}\, dy = 0 \text{ for all } \boldsymbol{T} = (T_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ satisfying } \operatorname{div} \mathbf{div}\, \boldsymbol{T} = 0 \text{ in } H^{-2}(\omega),\\ &\int_\omega \{E_{\alpha\beta}S_{\alpha\beta} + (\det \boldsymbol{F})\Phi(\boldsymbol{S})\}\, dy = 0 \text{ for all } \boldsymbol{S} = (S_{\alpha\beta}) \in \mathbb{L}^2(\omega) \text{ satisfying } \mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\omega)\end{aligned}$$

where Φ is the mapping defined in Theorem 3.1. *Then the set $\mathbb{T}(\omega)$ is sequentially weakly closed in the space $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$.*

(b) *Given any $((E_{\alpha\beta}, F_{\alpha\beta})) \in \mathbb{T}(\omega)$, there exists by* Theorem 3.1 *a unique vector field $\boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ such that*

$$\begin{aligned}\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) &= E_{\alpha\beta} \text{ in } L^2(\omega),\\ \partial_{\alpha\beta}\eta_3 &= F_{\alpha\beta} \text{ in } L^2(\omega).\end{aligned}$$

Let $\boldsymbol{F} : (\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{T}(\omega) \to \boldsymbol{\eta} \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ denote the mapping defined in this fashion. Then $\boldsymbol{F}$ maps weakly convergent sequences in the set $\mathbb{T}(\omega)$ into sequences that strongly converge in the space $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ endowed with the norm of the space $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$.

Proof. (i) The proof rests on the following *nonlinear Korn's inequality*: there exists a constant C such that

$$\begin{aligned}\big\|\boldsymbol{\eta}\big\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} \leq C \Big(&\|\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3)\|_{\mathbb{L}^2(\omega)}\\ &+ \|(\partial_{\alpha\beta}\eta_3)\|_{\mathbb{L}^2(\omega)} + \|(\partial_{\alpha\beta}\eta_3)\|^2_{\mathbb{L}^2(\omega)}\Big)\end{aligned}$$

for all $\boldsymbol{\eta} = (\eta_1) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$. To prove this inequality, it suffices to use the two-dimensional Korn's inequality for vector fields $(\eta_\alpha) \in H_0^1(\omega) \times H_0^1(\omega)$, the inequality

$$\|(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha)\|_{L^2(\omega)} \leq \|\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3\|_{L^2(\omega)} + \|\partial_\alpha\eta_3\partial_\beta\eta_3\|_{L^2(\omega)},$$

and the continuous injection $H^1(\omega) \hookrightarrow L^4(\omega)$ (which holds since ω is a two-dimensional domain), the combination of which implies that there exists a constant c such that

$$\|\partial_\alpha\eta_3\partial_\beta\eta_3\|_{L^2(\omega)} \le \|\partial_\alpha\eta_3\|_{L^4(\omega)}\,\|\partial_\beta\eta_3\|_{L^4(\omega)} \le c\,\|\partial_\beta\eta_3\|^2_{H^1(\omega)} \le c\,\|\eta_3\|^2_{H^2(\omega)}.$$

Since the mapping $\boldsymbol{F} : \mathbb{T}(\omega) \to H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$ is one-to-one and onto by Theorem 3.1, the above Korn's inequality can be immediately converted into an *inequality for elements in the set* $\mathbb{T}(\omega)$, viz.,

$$\|\boldsymbol{F}(\boldsymbol{E},\boldsymbol{F})\|_{H^1(\omega)\times H^1(\omega)\times H^2(\omega)} \le C\left(\|\boldsymbol{E}\|_{\mathbb{L}^2(\omega)} + \|\boldsymbol{F}\|_{\mathbb{L}^2(\omega)} + \|\boldsymbol{F}\|^2_{\mathbb{L}^2(\omega)}\right)$$

for all $(\boldsymbol{E},\boldsymbol{F}) \in \mathbb{T}(\omega)$.

(ii) Let $((\boldsymbol{E}^k,\boldsymbol{F}^k))_{k=1}^\infty$ be a sequence of elements $(\boldsymbol{E}^k,\boldsymbol{F}^k) \in \mathbb{T}(\omega)$ that weakly converges to $(\boldsymbol{E},\boldsymbol{F}) \in \mathbb{L}^2(\omega)\times\mathbb{L}^2(\omega)$. Since this sequence is then bounded in $\mathbb{L}^2(\omega)\times\mathbb{L}^2(\omega)$, the sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$, where $\boldsymbol{\eta}^k := \boldsymbol{F}(\boldsymbol{E}^k,\boldsymbol{F}^k)$, is bounded in the space $H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$ by (i). Hence there exist a subsequence $(\boldsymbol{\eta}^\ell)_{\ell=1}^\infty$ and $\boldsymbol{\eta} \in H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$ such that

$$\boldsymbol{\eta}^\ell \rightharpoonup \boldsymbol{\eta} \text{ in } H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega) \text{ as } \ell\to\infty,$$

where $\rightharpoonup$ denotes weak convergence. Therefore,

$$\boldsymbol{\eta}^\ell \to \boldsymbol{\eta} \text{ in } L^2(\omega)\times L^2(\omega)\times H^1(\omega) \text{ as } \ell\to\infty,$$

thanks to the compact injections $H^1(\omega) \Subset L^2(\omega)$ and $H^2(\omega) \Subset H^1(\omega)$ (which hold since ω is a two-dimensional domain). From these properties, it is then easy to conclude that $\boldsymbol{\eta} = \boldsymbol{F}(\boldsymbol{E},\boldsymbol{F})$, hence that $(\boldsymbol{E},\boldsymbol{F}) \in \mathbb{T}(\omega)$. Consequently, the set $\mathbb{T}(\omega)$ is *sequentially weakly closed* in $\mathbb{L}^2(\omega)\times\mathbb{L}^2(\omega)$.

Finally, the uniqueness of the limit shows that the whole sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$ converges strongly to $\boldsymbol{\eta}$ in the space $L^2(\omega)\times L^2(\omega)\times H^1(\omega)$. □

Thanks to Theorem 4.1, the minimization problem of the nonlinear Kirchhoff-von Kármán-Love theory when $\gamma_0 = \gamma$, i.e., with the displacement field $\boldsymbol{\eta}^* = (\eta_i^*) \in H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$ as the unknown (Section 1), can now be recast as one of the **intrinsic approach** to this theory, i.e., as a minimization problem with

$$E^*_{\alpha\beta} := \frac{1}{2}(\partial_\alpha\eta^*_\beta + \partial_\beta\eta^*_\alpha + \partial_\alpha\eta^*_3\partial_\beta\eta^*_3) \in L^2(\omega) \text{ and } F^*_{\alpha\beta} := \partial_{\alpha\beta}\eta^*_3 \in L^2(\omega)$$

as the new unknowns. This is the object of the next theorem, where the existence and uniqueness of the solution to this new minimization problem are also established; this result constitutes the *second main result* of this paper.

Recall that the constants $a_{\alpha\beta\sigma\tau}$ are the components of the two-dimensional elasticity tensor of the plate and that the functions $p_i \in L^2(\omega)$ and $q_\alpha \in L^2(\omega)$ designate the resultants and couples acting on the plate (Section 1).

Theorem 4.2. *Let the space* $\mathbb{T}(\omega)$ *and the mapping* $\boldsymbol{F} : \mathbb{T}(\omega) \to H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega)$ *be defined as in* Theorem 4.1 *and let the functional* $I : \mathbb{T}(\omega) \to \mathbb{R}$ *be defined for each* $(\boldsymbol{E},\boldsymbol{F}) = ((E_{\alpha\beta}),(F_{\alpha\beta})) \in \mathbb{T}(\omega)$ *by*

$$I(\boldsymbol{E},\boldsymbol{F}) := \frac{1}{2}\int_\omega\left\{\varepsilon a_{\alpha\beta\sigma\tau}E_{\sigma\tau}E_{\alpha\beta} + \frac{\varepsilon^3}{3}a_{\alpha\beta\sigma\tau}F_{\sigma\tau}F_{\alpha\beta}\right\}\mathrm{d}y - L(\boldsymbol{F}(\boldsymbol{E},\boldsymbol{F}))$$

where

$$L(\boldsymbol{\eta}) := \int_\omega p_i\eta_i\,\mathrm{d}y - \int_\omega q_\alpha\partial_\alpha\eta_3\,\mathrm{d}y \text{ for each } \boldsymbol{\eta} = (\eta_i) \in H^1_0(\omega)\times H^1_0(\omega)\times H^2_0(\omega).$$

Then, if the norms $\|p_\alpha\|_{L^2(\omega)}$ *are small enough, there exists at least one element* $(\boldsymbol{E}^*,\boldsymbol{F}^*) = ((E^*_{\alpha\beta}),(F^*_{\alpha\beta})) \in \mathbb{T}(\omega)$ *such that*

$$I(\boldsymbol{E}^*,\boldsymbol{F}^*) = \inf_{(\boldsymbol{E},\boldsymbol{F})\in\mathbb{T}(\omega)} I(\boldsymbol{E},\boldsymbol{F}).$$

Besides,

$$E^*_{\alpha\beta} := \frac{1}{2}(\partial_\alpha\eta^*_\beta + \partial_\beta\eta^*_\alpha + \partial_\alpha\eta^*_3\partial_\beta\eta^*_3) \in L^2(\omega) \text{ and } F^*_{\alpha\beta} := \partial_{\alpha\beta}\eta^*_3 \in L^2(\omega),$$

where the vector field $\boldsymbol{\eta}^* \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ *satisfies*

$$J(\boldsymbol{\eta}^*) = \inf_{\boldsymbol{\eta} \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)} J(\boldsymbol{\eta}),$$

and the functional $J : H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega) \to \mathbb{R}$ *is defined for each* $\boldsymbol{\eta} = (\eta_i) \in H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ *by*

$$J(\boldsymbol{\eta}) := \frac{1}{2} \int_\omega \left\{ \frac{\varepsilon}{4} a_{\alpha\beta\sigma\tau} (\partial_\sigma \eta_\tau + \partial_\tau \eta_\sigma + \partial_\sigma \eta_3 \partial_\tau \eta_3)(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) + \frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \eta_3 \partial_{\alpha\beta} \eta_3 \right\} \mathrm{d}y - L(\boldsymbol{\eta}).$$

Proof. (i) The function $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{T}(\omega) \to \dfrac{1}{2} \displaystyle\int_\omega \left\{ \varepsilon a_{\alpha\beta\sigma\tau} E_{\sigma\tau} E_{\alpha\beta} + \frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau} F_{\sigma\tau} F_{\alpha\beta} \right\} \mathrm{d}y$, which is clearly continuous, and convex since the assumed inequalities $\lambda \geq 0$ and $\mu > 0$ imply the existence of a constant $\alpha > 0$ such that

$$a_{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \geq \alpha t_{\alpha\beta} t_{\alpha\beta} \text{ for all } 2 \times 2 \text{ symmetric matrices } (t_{\alpha\beta}),$$

is therefore sequentially weakly lower semi-continuous (see, e.g., Theorem 9.2-3 in [4]).

The function $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{T}(\omega) \to L(\boldsymbol{F}(\boldsymbol{E}, \boldsymbol{F})) \in \mathbb{R}$ is sequentially weakly continuous since, by Theorem 4.1,

$$(\boldsymbol{E}^k, \boldsymbol{F}^k) \underset{k \to \infty}{\rightharpoonup} (\boldsymbol{E}, \boldsymbol{F}) \text{ in } \mathbb{T}(\omega) \text{ implies } \boldsymbol{F}(\boldsymbol{E}^k, \boldsymbol{F}^k) \underset{k \to \infty}{\to} \boldsymbol{F}(\boldsymbol{E}, \boldsymbol{F}) \text{ in } L^2(\omega) \times L^2(\omega) \times H^1(\omega),$$

and the linear form $L : L^2(\omega) \times L^2(\omega) \times H^1(\omega) \to \mathbb{R}$ is continuous.

Consequently, *the functional* $I : \mathbb{T}(\omega) \to \mathbb{R}$ *is sequentially weakly lower semi-continuous*.

(ii) In what follows, the constants C_1, C_2, and C_3 are independent of the various functions, vector fields, or tensor fields, appearing in a given inequality. First, it is clear that

$$\begin{aligned} I(\boldsymbol{E}, \boldsymbol{F}) \geq & C_1 \left(\|\boldsymbol{E}\|^2_{\mathbb{L}^2(\omega)} + \|\boldsymbol{F}\|^2_{\mathbb{L}^2(\omega)} \right) - \|(p_\alpha)\|_{\boldsymbol{L}^2(\omega)} \|(\eta_\alpha)\|_{\boldsymbol{L}^2(\omega)} \\ & - \|p_3\|_{L^2(\omega)} \|\eta_3\|_{L^2(\omega)} - \|(q_\alpha)\|_{\boldsymbol{L}^2(\omega)} \|(\partial_\alpha \eta_3)\|_{\boldsymbol{L}^2(\omega)}, \end{aligned}$$

for all $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{T}(\omega)$, where $\boldsymbol{\eta} = \boldsymbol{F}(\boldsymbol{E}, \boldsymbol{F})$. Second, as already noted in the proof of Theorem 4.1,

$$\|(\eta_\alpha)\|_{\boldsymbol{L}^2(\omega)} \leq \|(\eta_\alpha)\|_{\boldsymbol{H}^1(\omega)} \leq C_2 \left(\|\boldsymbol{E}\|_{\mathbb{L}^2(\omega)} + \|\boldsymbol{F}\|^2_{\mathbb{L}^2(\omega)} \right),$$

so that

$$\begin{aligned} I(\boldsymbol{E}, \boldsymbol{F}) \geq & C_1 \ \|\boldsymbol{E}\|^2_{\mathbb{L}^2(\omega)} + \left(C_1 - C_2 \|(p_\alpha)\|_{\boldsymbol{L}^2(\omega)} \right) \|\boldsymbol{F}\|^2_{\mathbb{L}^2(\omega)} \\ & - C_2 \|(p_\alpha)\|_{\boldsymbol{L}^2(\omega)} \|\boldsymbol{E}\|_{\mathbb{L}^2(\omega)} - C_3 \left(\|(p_3)\|_{\boldsymbol{L}^2(\omega)} + \|(q_\alpha)\|_{\boldsymbol{L}^2(\omega)} \right) \|\boldsymbol{F}\|_{\mathbb{L}^2(\omega)} \end{aligned}$$

for all $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{T}(\omega)$. Hence *the functional* $I : \mathbb{T}(\omega) \to \mathbb{R}$ *is coercive if the norms* $\|p_\alpha\|_{L^2(\omega)}$ *are small enough.*

(iii) By a standard result from the calculus of variations (see, e.g., Theorem 3.30 in Dacorogna [10] or Theorem 9.3-1 in [4]), there thus exists at least one minimizer $(\boldsymbol{E}^*, \boldsymbol{F}^*)$ of the functional I in the set $\mathbb{T}(\omega)$.

(iv) Given such a minimizer $(\boldsymbol{E}^*, \boldsymbol{F}^*) \in \mathbb{T}(\omega)$, the definitions of the functionals I and J clearly imply that the vector field $\boldsymbol{F}(\boldsymbol{E}^*, \boldsymbol{F}^*)$ minimizes the functional J over the space $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$. □

Note that a similar *intrinsic approach* could be *a fortiori* applied to the Kirchhoff-von Kármán-Love theory for a *linearly elastic plate*, i.e., where the change of metric tensor $\left(\frac{1}{2} (\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \right)$ is replaced by the *linearized change of metric tensor* $\left(\frac{1}{2} (\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) \right)$. In this case, the nonlinear Donati compatibility conditions of Theorem 3.1 are to be replaced by their linearized version found in Theorem 2.4.

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